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Free Subgroups of Free Complete Products

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1. SUMMARY

Recently the existence of free subgroups of free complete products of the integer group $\mathbb{Z}$ of large cardinalities have been shown by Zastrow [8] and by Cannon and Conner [1]. The two subgroups are very different subgroups, though they are isomorphic free groups. (Short proofs of these results are given by Eda [3].) In the present paper we generalize the results to the case for free complete products of arbitrary groups. It turns out that what changes is a part of a free product $*_{i\in I} G_i$ and the remaining part is free, as in the case of copies of $\mathbb{Z}$, but does not reflect particular structures of groups G_i's. The undefined notion will be stated in Sections 3 and 4.

THEOREM 1.1. *Let Bd be a subgroup of the free complete product* $\mathbb{X}_{i\in I} G_i$ *consisting of elements of bounded appearance and let Sc be a subgroup consisting of elements whose corresponding reduced words have scattered order types. Then both Bd and Sc are isomorphic to* $*_{i\in I} G_i * *_\lambda \mathbb{Z}$, *where* λ *is the cardinal determined by the following: Let* I_0 *be the subset of* I *such that* $i \in I_0$

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if and only if $\{j \in I\colon |G_j| \geq |G_i|\}$ *is infinite and let* λ *be the cardinality of* $\prod_{i\in I_0} G_i$, *which is equal to that of* $\mathbb{X}_{i\in I_0} G_i$.

COROLLARY 1.2. *Let* I *be an infinite index set and* G_i*'s be copies of a group* G. *Then the subgroups of* Bd *and* Sc *of* $\mathbb{X}_{i\in I} G_i$ *are isomorphic to* $*_{i\in I} G_i * *_{|G^I|}\mathbb{Z}$.

S. Shelah *et al.* [4, 7] showed that the Specker phenomenon, i.e., the Higman theorem [5], behaves differently in the uncountable case than in the countable case (we explain this more precisely in Remark 5.1). We shall show the existence of large free retracts of uncountable free complete products by modifying the proof in [7] as follows. Since we use the Dedekind cuts explicitly, our presentation of the proof is shorter than that in [7].

THEOREM 1.3. *Let* G_i *be a nontrivial group for each* $i \in I$ *and let* I_1 *be the subset of* I *such that* $i \in I_1$ *if and only if* $\{j \in I\colon |G_j| \geq |G_i|\}$ *is uncountable. Then there exists a free retract of the free complete product* $\mathbb{X}_{i\in I} G_i$ *whose cardinality is the same as that of* $\mathbb{X}_{i\in I_1} G_i$.

Sometimes we use set theoretical notation. An ordinal is the set of all ordinals strictly smaller than itself and a cardinal is an initial ordinal. The uncountable cofinality is essential in the proof of Theorem 1.3. We refer the reader to [6] for elementary notions of ordinals and cardinals. The proofs of the above theorems involve cardinality arguments, which are elementary but belong to set theory. We collect them in the last section.

REMARK 1.4. (1) The set $I\backslash I_0 = \{i \in I\colon i \notin I_0\}$ is finite and $\mathbb{X}_{i\in I} G_i = *_{i\in I\backslash I_0} G_i * \mathbb{X}_{i\in I_0} G_i$ in Theorem 1.1. In Theorem 1.3 the set $I\backslash I_1$ is countable, but $\mathbb{X}_{i\in I} G_i$ is not isomorphic to $\mathbb{X}_{i\in I\backslash I_1} G_i * \mathbb{X}_{i\in I_1} G_i$ in general. Particularly if G_i's are copies of a nontrivial group G and I is uncountable, the cardinalities of $\mathbb{X}_{i\in I_0} G_i$ and $\mathbb{X}_{i\in I_1} G_i$ are equal to the cardinality of the whole group $\mathbb{X}_{i\in I} G_i$.

(2) The free subgroups in Theorem 1.1 are not retracts. This can be seen as follows. The restriction map from I to a countable subset J induces a homomorphism from $\mathbb{X}_{i\in I} G_i$ to a subgroup $\mathbb{X}_{i\in J} G_i$. By this homomorphism, Sc and Bd are mapped precisely onto the same as those defined for $\mathbb{X}_{i\in J} G_i$. But, by an application of the Higman theorem a homomorphic image of $\mathbb{X}_{i\in J} G_i$ to a free group is equal to an image of $*_{i\in X} G_i$ for some finite subset X of J. Therefore free subgroups in Theorem 1.1 are not retracts.

2. DEFINITIONS AND PRELIMINARY FACTS

We follow the notation in [2] basically, but state definitions and lemmas for the reader's convenience.

DEFINITION 2.1. For groups G_i $(i \in I)$, suppose that $G_i \cap G_j = \{e\}$. A word $W \in \mathcal{W}(G_i\colon i \in I)$ is a function from a linearly ordered set $\overline{W}$ to $\bigcup_{i\in I} G_i$ such that $\{\alpha \in \overline{W}\colon W(\alpha) \in G_i\}$ is finite for each $i \in I$.

$\overline{W}^- = \{\alpha^-\colon \alpha \in \overline{W}\}$ denotes the inversely ordered set of $\overline{W}$, where $\alpha^- < \beta^-$ if $\beta < \alpha$. W^{-1} is a word such that $\overline{W^{-1}} = \overline{W}^-$ and $W^{-1}(\alpha^-) = W(\alpha)^{-1}$.

We identify two words V and W, if there exists an order preserving bijection $\varphi\colon \overline{V} \to \overline{W}$ such that $V(\alpha) = W(\varphi(\alpha))$ for each $\alpha \in \overline{V}$, for which we write $V \cong W$.

Let $p_{XY}\colon *_{i\in Y} G_i \to *_{i\in X} G_i$ be the projection for finite subsets $X \subseteq Y$ of I, where p_{XX} is the identity. The notation $X \Subset I$ means that X is a finite subset of I. Thus the unrestricted free product is $\varprojlim(*_{i\in X} G_i, p_{XY}\colon X \subseteq Y \Subset I)$ [5].

For a word $W \in \mathcal{W}(G_i, i \in I)$ and $X \Subset I$, W_X is the word obtained from W by restricting to letters in $\cup\{G_i\colon i \in X\}$; i.e., $\overline{W_X} = \{\alpha \in \overline{W}\colon W(\alpha) \in G_i \text{ for some } i \in X\}$ and $W_X = W \restriction \overline{W_X}$. W_X can be regarded as an element of the free product $*_{i\in X} G_i$. Two words V and W are equivalent, if $V_X = W_X$ as elements of the free product for any $X \Subset I$. The equivalence class containing W is denoted by $[W]$. Then we get a free complete product $\mathbb{X}_{i\in I} G_i = \{[W]\colon W \in \mathcal{W}(G_i\colon i \in I)\}$, where the operation is defined by the concatenation. This group is isomorphic to the subgroup $\bigcap_{F \Subset I}\{*_{i\in X} G_i * \varprojlim(*_{i\in X} G_i, p_{XY}\colon X \subseteq Y \Subset I\backslash F)\}$ of the unrestricted free product [2, Proposition 1.8].

DEFINITION 2.2. A word V is a *subword* of W, if there are words X and Y such that $W \cong XVY$. V is a *proper subword*, in case at least one of X and Y is not empty. V is an *initial subword*, in case X is empty. A word V is a *quasi-subword* of W; there exists a subword V' such that the following hold: $V \cong V'$, $V \cong gV'$, and $V \cong V'g'$, or $V \cong gV'g'$ holds for some letters g, g'.

A word W is *reduced*, if $W \cong UXV$ implies $[X] \neq e$ for any nonempty word X and for any neighboring elements α and β of $\overline{W}$. It never occurs that $W(\alpha)$ and $W(\beta)$ belong to the same G_i. A word W is *quasi-reduced*, if $W \cong UXV$ with $[X] = e$ implies $\mathrm{Im}(X) \subseteq G_i$ for some i and the existence of $e \neq g \in G_i$ for some i such that g is the rightmost letter of U or the leftmost letter of V.

In other words a reduced word can be obtained from a quasi-reduced word by multiplying neighboring elements.

PROPOSITION 2.3 ([2, Theorem 1.4]). *For each word $W \in \mathcal{W}(G_i\colon i \in I)$ there exists a unique reduced word $V \in \mathcal{W}(G_i\colon i \in I)$ such that $[V] = [W]$.*

We note that the notation "$\simeq$" in [2] is the same as our "$\cong$."

PROPOSITION 2.4 ([2, Corollary 1.7]). *Let V and W be reduced words. Then there exist reduced words X, Y, Z such that $V \cong XY$ and $W \cong Y^{-1}Z$ hold and XZ is quasi-reduced.*

We easily have

LEMMA 2.5. *For groups G_{ij} $i \in \{0, 1\}$, $j \in J$, $\mathbb{X}_{(i,j)\in I\times\{0,1\}} G_{ij} \simeq \mathbb{X}_{i\in I} G_{i0} * G_{i1}$.*

3. THE SUBGROUP OF BOUNDED APPEARANCE

Definitions and lemmas in Sections 3 and 4 are similar to those in [3]. We remark that even in case G_i's are copies of $\mathbb{Z}$ the notion "bounded appearance" is a weaker condition than that in [3] and consequently the resulting subgroup Bd is larger than Bd in [3].

DEFINITION 3.1. For $i \in I$ and a word $W \in \mathscr{W}(G_i\colon i \in I)$, $l_i(W)$ denotes the number of appearances of elements in G_i in W, i.e., the cardinality of $\{\alpha \in \overline{W}\colon W(\alpha) \in G_i\}$. A word W is of appearance n, if $\max\{l_i(W)\colon i \in I\} = n$. The subgroup of bounded appearance of $\mathbb{X}_{i\in I} G_i$ is $Bd = \{[W]\colon W \text{ is of appearance } n, n < \omega\}$, where ω is the least infinite ordinal and $n < \omega$ means that n is a natural number. Let Bd_n be the subgroup of Bd generated by words of appearance less than or equal to n.

A subgroup G of $\mathbb{X}_{i\in I} G_i$ is *fine*, if the following hold:

- G contains $*_{i\in I} G_i$;
- If V is a subword of a reduced word W and $[W] \in G$ holds, $[V] \in G$ also holds.

For a subset X of a group G, $\langle X\rangle$ denotes the subgroup of G generated by X. Following the convention for words of finite length, we use the expression "$W = V$" instead of $[W] = [V]$ when no confusion will occur. We also write $W \in G$ for a group G instead of $[W] \in G$. We remark that $W \cong V$ implies $W = V$, but the converse does not hold in general.

LEMMA 3.2. *Let G be a fine subgroup of $\mathbb{X}_{i\in I} G_i$ and W a reduced word. Then $\langle G \cup \{V\colon$ an initial subword of $W\}\rangle$ is fine.*

Proof. Let $X_0, \ldots, X_m$ be reduced words which belong to G or initial subwords of W. It can be easily proved by induction on m that any initial subword of the reduced word of $X_0 \ldots X_m$ belongs to $\langle G \cup \{V\colon$ an initial subword of W$\}\rangle$. Since any subword is a product of the inverse of an initial subword and another initial subword, we get the conclusion. ∎

LEMMA 3.3. *Let $A_0, \ldots, A_n$ and let W be reduced words such that $W = A_0 \cdots A_n$. Then there exist reduced words $W_0, \ldots, W_m$ and $0 \leq i_0 < \cdots < i_m \leq n$ satisfying the following*:

- *$W = W_0 \cdots W_m$ and $W_0 \cdots W_m$ is quasi-reduced*;
- *each W_k is a subword of A_{i_k}, which is presented in the next equation* (*in other words each W_k is unchanged in the reduction indicated in the next item*);
- *$A_0 \cdots A_n \cong X_0 W_0 \cdots X_m W_m X_{m+1}$ for some words X_i such that $X_i = e$ for each i.*

Proof. The proof is by induction on n. Since the case $n = 0$ follows from Proposition 2.3, we prove the induction steps. Let A be the reduced word of $A_0 \cdots A_{n-1}$. Then $W = AA_n$ holds and hence there exist reduced words X, B, and C such that $A \cong BX$, $A_n \cong X^{-1}C$, and BC is quasi-reduced and $W = BC$ and by Proposition 2.4. Apply the induction hypothesis to A and let $V_0, \ldots, V_k$ be the obtained reduced words. Now $B = V_0 \cdots V_{j-1} V_j'$ and $V_0 \cdots V_{j-1} V_j'$ is quasi-reduced, where V_j' is an initial subword of V_j. When V_j is a subword of A_{i_j}, V_j' is also a subword of A_{i_j}. Hence $W = V_0 \cdots V_{i-1} V_i' C$ holds and $V_0, \ldots, V_{i-1}, V_i', C$ satisfy the required properties. ∎

Now we start to prove lemmas for the first half of Theorem 1.1.

LEMMA 3.4. *Let W be a word of appearance n and let a subword A of W be also of appearance n. If $W \cong X_0 A Y_0 \cong X_1 A Y_1$ holds, then $X_0 \cong X_1$ and $Y_0 \cong Y_1$ hold.*

Proof. Let $l_i(A) = n$ and consider the leftmost appearance of a letter belonging G_i and the rightmost appearance. Then we can easily get the conclusion. ∎

LEMMA 3.5. *Let G be a fine subgroup of Bd_n containing Bd_{n-1} and let A and W be reduced words of appearance n such that W is a subword of A and $W \notin G$. Then W is not a subword of A^{-1}.*

Proof. We first remark that $|\{\alpha\colon W(\alpha) \in G_i\}| = n$ for infinitely many i's by the assumption.

Suppose that W is a subword of A^{-1}. Since both A and W are of appearance n, we have a subword V of W of appearance n such that $V \cong V^{-1}$ and $V \notin G$. By [2, Corollary 1.6], $V \cong U^{-1} g U$ with $g^2 = e$. Since G contains $*_{i \in I} G_i$ and Bd_{n-1}, g and U belong to G, which contradicts $V \notin G$. ∎

LEMMA 3.6. *Let G be a fine subgroup of Bd_n containing Bd_{n-1} and let $A_0, \ldots, A_m$ be reduced words of appearance n such that $\{i\colon l_i(A_k) = n\} \cap \{i\colon l_i(A_{k'}) = n\} = \emptyset$ for distinct k and k'. Suppose that $A_i \notin G$ for any $0 \leq i \leq m$. Then $\langle G \cup \{A_i\colon 0 \leq i \leq m\} \rangle = G * *_{i=0}^m \langle A_i \rangle$ holds.*

Proof. Suppose the contrary. There exist $Y_0 \cdots Y_k = e$ such that each Y_i is reduced and the left term is a nonempty reduced form in $G * *_{i=0}^{m}\langle A_i \rangle$; i.e., the letters are elements of G, A_i's, and A_i^{-1}'s. We assume that k is the minimum in such numbers. Since we may assume one of Y_i's is A_m or A_m^{-1}, we may also assume that Y_0 is A_m. Now apply Lemma 3.3 to $A_m = Y_k^{-1} \cdots Y_1^{-1}$. We have reduced quasi-subwords $W_0, \ldots, W_s$ of A_m and $X_0, \ldots, X_{s+1}$ with the properties in Lemma 3.3. If W_j is of appearance n, W_j cannot be a subword of any A_i for $0 \leq i < m$. Since $A_m \notin G$ and G is fine and $G \supset Bd_{n-1}$, there exist W_{j_0} and Y_{k_0} such that

- W_{j_0} is of appearance n and a quasi-subword of $Y_{k_0}^{-1}$;
- $W_{j_0} \notin G$;
- $Y_{k_0} \cong A_m$ or $Y_{k_0} \cong A_m^{-1}$.

By Lemma 3.5, we have $Y_{k_0}^{-1} \cong A_m$. We observe the following equation:

$$\begin{aligned} &Y_k^{-1} \cdots Y_{k_0}^{-1} W_0 \cdots W_{j_0} \cdots W_s Y_{k_0-1}^{-1} \cdots Y_1^{-1} \\ &= Y_k^{-1} \cdots Y_{k_0}^{-1} A_m Y_{k_0-1}^{-1} \cdots Y_1^{-1} \\ &\cong X_0 W_0 X_1 \cdots X_{j_0} W_{j_0} X_{j_0+1} \cdots W_s X_{s+1}. \end{aligned}$$

By Lemmas 3.4 and 3.5 the word W_{j_0} in the first term corresponds to W_{j_0} in the third. More precisely a subword W'_{j_0} of A_m of appearance n for which one of $W_{j_0} \cong W'_{j_0}$, $W_{j_0} \cong gW'_{j_0}$, $W_{j_0} \cong W'_{j_0}g'$, or $W_{j_0} \cong gW'_{j_0}g'$ for some letters g, g' in the first term is precisely W'_{j_0} in W_{j_0} in the third. Consequently $W_{j_0+1} \cdots W_s Y_{k_0-1}^{-1} \cdots Y_1^{-1} = X_{j_0+1} W_{j_0+1} \cdots W_s X_{s+1}$. Therefore both of $Y_{k_0-1}^{-1} \cdots Y_1^{-1} = e$ and $Y_k^{-1} \cdots Y_{k_0+1}^{-1} = e$ hold, which contradicts the minimality of k. ∎

The next lemma is well known and easy to prove.

LEMMA 3.7. *Let F be a free group generated by $e_1, \ldots, e_n$. Then F is freely generated also by $e_1, e_1 e_2, \ldots, e_1 e_2 \cdots e_n$.*

Proof (first half of Theorem 1.1). We shall prove that Bd_1 is the free product of $*_{i \in I} G_i$ and a free group and Bd_n is the free product of Bd_{n-1} and a free group for $n \geq 2$ by induction. The cardinality argument concerning λ will be settled in Sections 4 and 6. We remark that Bd_0 is a trivial group.

The initial stage: set $G = *_{i \in I} G_i$. Our induction hypothesis is the following: G is a fine subgroup of Bd_n containing Bd_{n-1} as a free factor. If $G = Bd_n$ we have finished the nth step. Otherwise, there exists a reduced word W of appearance n such that $W \notin G$. Apply Lemma 3.6 for $i_0 = m = 0$, then we conclude $\langle G \cup \{W\} \rangle = G * \langle W \rangle$. We pick initial subwords of W as generators inductively. Suppose that $\mathcal{V} \subseteq \{V\colon \text{an initial subword of } W\}$ freely generates $\langle G \cup \mathcal{V} \rangle$ with G; i.e., $\langle G \cup \mathcal{V} \rangle = G * *_{V \in \mathcal{V}} \langle V \rangle$. If

$\langle G \cup \mathcal{V} \rangle = \langle G \cup \{V\colon \text{ an initial subword of } W\}\rangle$, then $G * *_{V \in \mathcal{V}} \langle V \rangle = \langle G \cup \{V\colon \text{ an initial subword of } W\}\rangle$ is a fine subgroup by Lemma 3.2 and we proceed to the procedure to find a reduced word of appearance n outside of the constructed subgroup. Otherwise, there exists an initial subword U of W such that $U \notin \langle G \cup \mathcal{V} \rangle$. We claim that $\langle G \cup \mathcal{V} \cup \{U\}\rangle = G * *_{V \in \mathcal{V}} \langle V \rangle * \langle U \rangle$. To see this, let $V_0, \ldots, V_m \in \mathcal{V}$ so that V_j is an initial subword of V_{j+1} for $0 \le j \le m-1$, V_i is an initial subword of U, U is an initial subword of V_{i+1}, and $V_m \cong W$. Let A_j be reduced words such that

$$A_j = \begin{cases} V_0, & \text{for } j = 0; \\ V_{j-1}^{-1} V_j, & \text{for } 0 < j \le i; \\ V_i^{-1} U, & \text{for } j = i+1; \\ U^{-1} V_{i+1}, & \text{for } j = i+2; \\ V_{j-2}^{-1} V_{j-1}, & \text{for } i+2 < j \le m+1. \end{cases}$$

The facts $A_j \notin G$ for $j \le i$ and $j \ge i+3$ follow from the fact that $\langle G \cup \mathcal{V} \rangle = G * *_{V \in \mathcal{V}} \langle V \rangle$. The facts $A_{i+1} \notin G$ and $A_{i+2} \notin G$ follow from $U \notin \langle G \cup \{V_i, V_{i+1}\}\rangle$. Then $A_0 \cdots A_{m+1} \cong W$ holds and hence $\langle G \cup \{A_i\colon 0 \le i \le m+1\}\rangle = G * *_{i=0}^{m+1} \langle A_i \rangle$ by Lemma 3.6. Now $\langle G \cup \{V_i\colon 0 \le i \le m\} \cup \{U\}\rangle = G * *_{i=0}^{m+1} \langle A_i \rangle = G * *_{i=0}^{m} \langle V_i \rangle * \langle U \rangle$ by Lemma 3.7, which implies the claim. Just iterating the above process transfinitely, we get the desired bases for Bd_n. ∎

4. THE SUBGROUP OF SCATTERED TYPE

First we recall scattered sets. For a space X, X' denotes the subspace of X consisting of all nonisolated points. Let $X_0 = X$ and $X_{\alpha+1} = X'_\alpha$ for an ordinal α and let $X_\alpha = \cap\{X_\beta\colon \beta < \alpha\}$ for a limit ordinal α. A space is said to be *scattered*, if X_α is empty for some α. Since a linearly ordered set can be regarded as a topological space with its order topology, we call a linearly ordered set scattered in case it is scattered under its order topology. For a linearly ordered set L, let $D(L)$ be the set of the Dedekind cuts of L, which becomes a linearly ordered set. Words V and W are *tail-equivalent*, if there exists a nonempty word X such that $V \cong YX$ and $W \cong ZX$ for some words Y and Z.

Let Sc_α be the subgroup of Sc generated by all W's such that W are reduced and $D(\overline{W})_\alpha = \emptyset$. Then Sc_1 is a subgroup consisting of all words of finite length; i.e., $*_{i \in I} G_i$. For an ordinal $\beta \ge 1$, let $\mathcal{V}_0(\beta)$ be the set of all reduced words W such that $D(\overline{W})_\beta$ is a singleton consisting of the largest element of $D(\overline{W})$. Choose a representative from each tail-equivalent class of $\mathcal{V}_0(\beta)$ and let $\mathcal{V}_1(\beta)$ be the set of such representatives. For $V \in \mathcal{V}_1(\beta)$, an *essential part* of V is a tail part of V, i.e., a nonempty subword W of V

such that $V \cong XW$ for some X. An essential part of V^{-1} is a head part of V^{-1}, i.e., a nonempty subword W of V^{-1} such that $V^{-1} \cong WX$ for some X.

LEMMA 4.1. *Let W be a reduced word with $W \in Sc_\beta$ and $V \in \mathscr{V}_1(\beta)$. Then VW and WV^{-1} are reduced words.*

Proof. Suppose that VW is not reduced. There is a tail X of V such that X^{-1} is a head of W by Proposition 2.3. This implies $W \notin Sc_\beta$ by the definition of $\mathscr{V}_1(\beta)$. The other is similarly proved. ∎

LEMMA 4.2. *Let W be a nonempty reduced word with $W \in Sc_\beta$, $V_0, V_1 \in \mathscr{V}_1(\beta)$, and $\varepsilon_0, \varepsilon_1 = \pm 1$. If X is the reduced word of $V_0^{\varepsilon_0} W V_1^{\varepsilon_1}$, then the essential parts of $V_0^{\varepsilon_0}$ and $V_1^{\varepsilon_1}$ remain in X.*

Proof. In the case of $\varepsilon_0\varepsilon_1 = 1$, at least one of $V_0^{\varepsilon_0}W$ and $WV_1^{\varepsilon_1}$ is reduced and we conclude the essential part of $V_0^{\varepsilon_0}$ and $V_1^{\varepsilon_1}$ remain in X. In the case of $\varepsilon_0 = 1$ and $\varepsilon_1 = -1$, V_0WV_1 is reduced by Lemma 4.1 and the conclusion is clear. In the remaining case—i.e., $\varepsilon_0 = -1$ and $\varepsilon_1 = 1$—the reduced word V of WV_1 belongs to $\mathscr{V}_0(\beta)$. Suppose that the essential part of V is cancelled in the reduced word of $V_0^{-1}V$. Then V_0 and V are tail-equivalent by the fact $V_0, V \in \mathscr{V}_0(\beta)$, and Lemma 4.1. Since V and V_1 are tail-equivalent, V_0 and V_1 are the same. Since $V_0^{-1}WV_1 \neq e$, the tail and head parts of $V_0^{-1}WV_1 \neq e$ remain in its reduced word of $V_0^{-1}WV_1$ and hence the conclusion holds. ∎

LEMMA 4.3. *Let W be the reduced word of $W_0W_1 \cdots W_n$, where*

- *$e \neq W_i \in Sc_\beta$ or $W_i \in \mathscr{V}_1(\beta) \cup \{V^{-1}: V \in \mathscr{V}_1(\beta)\}$ for each i;*
- *$W_i \in Sc_\beta$ if and only if $W_{i+1} \in \mathscr{V}_1(\beta) \cup \{V^{-1}: V \in \mathscr{V}_1(\beta)\}$ for each $0 \leq i \leq n-1$.*

Then the essential parts of $W_i \in \mathscr{V}_1(\beta) \cup \{V^{-1}: V \in \mathscr{V}_1(\beta)\}$ remain in W.

Proof. We prove this by induction on n. Let X be the reduced word of $W_1 \cdots W_n$. In the case of $W_0 \in Sc_\beta$, the essential part of W_1 remains in X by the induction hypothesis. Since the essential part of W_1 remains in the reduced word of W_0W_1, we get the conclusion by Lemma 4.1. In the other case—i.e., $W_0 \in \mathscr{V}_1(\beta) \cup \{V^{-1}: V \in \mathscr{V}_1(\beta)\}$—the essential part of W_2 remains in X and also in the reduced word of $W_0W_1W_2$ by Lemma 4.2. Therefore the conclusion follows from it. ∎

Before we prove the second half of Theorem 1.1, we construct free subgroups of $\mathbb{X}_{i\in I} G_i$ which are used in the proofs of Theorems 1.1 and 1.3. Let $G_{\alpha\beta}\,(\alpha < \kappa, \beta < \delta)$ be groups, let $a_{\alpha\gamma}\,(\alpha < \kappa, \gamma < \mu_\beta)$ be elements, and let $\mu_\beta\,(\beta < \delta)$ be cardinals such that

- $\mu_\beta \leq |G_{\alpha\beta} \backslash \{e\}|$ for every $\alpha < \kappa$;
- $\{a_{\alpha\gamma}: \gamma < \mu_\beta\} \subseteq G_{\alpha\beta} \backslash \{e\}$ and $a_{\alpha\gamma} \neq a_{\alpha\gamma'}$ for $\gamma \neq \gamma'$.

We regard that $\kappa \times \delta$ is lexicographically ordered.

LEMMA 4.4. *For $f \in \prod_{\beta<\delta} \mu_\beta$, let $\overline{W_f} = \kappa \times \delta$ and $W_f(\alpha, \beta) = a_{\alpha f(\beta)}$ for $\alpha < \kappa, \beta < \delta$. Then, W_f is a reduced word in $\mathscr{W}(G_{\alpha\beta}\colon \alpha < \kappa, \beta < \delta)$ and $\{W_f\colon f \in \prod_{\beta<\delta} \mu_\beta\}$ freely generates a subgroup; i.e., $\langle\{W_f\colon f \in \prod_{\beta<\delta} \mu_\beta\}\rangle = *_{f\in\prod_{\beta<\delta\mu_\beta}} \langle W_f\rangle$.*

Proof. Obviously each W_f is a reduced word in $\mathscr{W}(G_{\alpha\beta}\colon \alpha < \kappa, \beta < \delta\}$. Observe that $\{(\alpha, \beta)\colon \alpha < \kappa\}$ is cofinal in $\kappa \times \delta$ for each β. Let $f(\beta_0) \neq g(\beta_0)$ for $f, g \in \prod_{\beta<\delta\mu_\beta}$. Then $W_f(\alpha, \beta_0) \neq W_g(\alpha, \beta_0)$ for every $\alpha < \kappa$ and consequently W_f and W_g are not tail-equivalent for distinct f and g. $W_f W_g$ is reduced for any f and g and $W_f(W_g)^{-1}$ is reduced for distinct f and g. For distinct f and g, a cancellation of $(W_g)^{-1}W_f$ occurs between only $W_g(0, \gamma)^{-1}$'s and $W_f(0, \gamma)$'s and the head of $(W_g)^{-1}$ and the tail of W_f remain in the reduced word of $(W_g)^{-1}W_f$. Hence we have a conclusion. ∎

Proof (second half of Theorem 1.1). By induction on $\alpha \geq 1$, we prove that Sc_α factors to a free product of Sc_β and a free group for $1 \leq \beta < \alpha$ and Sc_α is free, which implies the theorem. Let α be a limit ordinal. By induction hypothesis, we choose a free subgroup T_γ of $Sc_{\gamma+1}$ so that $Sc_{\gamma+1} = Sc_\gamma * T_\gamma$ holds. Since $Sc_\gamma = \bigcup_{\delta<\gamma} Sc_\delta$ for a limit γ, $Sc_\alpha = Sc_1 * *_{1\leq\gamma<\alpha} T_\gamma$ holds and we get the conclusion. In case α is $\beta + 1$, it suffices to show that Sc_α factors to Sc_β and a free group and consequently Sc_α is free. Let L be a scattered compact linearly ordered set such that $L_\beta \neq \emptyset$ and $L_\alpha = \emptyset$. Then L_β is a finite set. Therefore Sc_α is generated by $Sc_\beta \cup \mathscr{V}_0(\beta)$. Suppose that words V and W in $\mathscr{V}_0(\beta)$ are tail-equivalent. There exist a nonempty word X and words Y and Z such that $V \cong YX$ and $W \cong ZX$ by definition. Then $Y, Z \in Sc_\beta$ and hence $V \in \langle Sc_\beta \cup \{W\}\rangle$. Therefore Sc_α is generated by Sc_β and $\mathscr{V}_1(\beta)$. Now $Sc_\alpha = Sc_\beta * *_{W\in\mathscr{V}_1(\beta)}\langle W\rangle$ by Lemma 4.3.

In the remaining part of this proof we show that the cardinality of the index set λ of two free subgroups of Bd and Sc is equal to $|\prod_{i\in I_0} G_i|$. By Lemma 2.5 we may assume that $|G_i| \geq 3$. Now we can apply Proposition 6.2 to the case that $X_i = G_i \backslash \{e\}$ for $i \in I_0$ and $\kappa = \omega$. Then we have $\{a_{\alpha\gamma}\colon \gamma < \mu_\beta\}$'s for the construction of W_f's. Now $[W_f]$ belongs to Bd_1, but does not belong to $*_{i\in I} G_i$. Moreover, if V is an initial subword of W_f, then $\overline{V}$ is an initial segment of $n \times \delta$ for some $n < \omega$. In the construction of Bd_1 we started from $*_{i\in I} G_i$ and then tried to find W of appearance 1 such that $[W] \notin G$. If we set W_f ($f \in \prod_{\beta<\delta} \mu_\beta$) at an initial part of a list of words of appearance 1, then the above observation about initial subwords of W_f's shows that W_f's are chosen as a free base in the proof. Therefore $\lambda \geq |\prod_{i\in I_0} G_i|$ by Lemma 4.4. On the other hand $\lambda \leq |\mathbb{X}_{i\in I_0} G_i| = |\prod_{i\in I_0} G_i|$ by Proposition 6.2 and we have $\lambda = |\prod_{i\in I_0} G_i|$ in the case of Bd. Next we think of the case of Sc. Since $W_f \in \mathscr{V}_0(1)$, we may choose W_f as an element of $\mathcal{V}_1(1)$ and can see $\lambda \geq |\prod_{i\in I_0} G_i|$. The remaining inequality follows from Proposition 6.3 as in the case of Bd. ∎

REMARK 4.5. We have shown that $Bd_1 \simeq *_{i\in I} G_i * *_\lambda \mathbb{Z}$. It holds that $Bd_{n+1} \simeq Bd_n * *_\lambda \mathbb{Z}$. This can be seen by a little modification of a construction, which we show here.

Let $V_f(\alpha, \beta) = (a_{(2\times\alpha)f(\beta)} a_{(2\times\alpha+1)f(\beta)})^{n+1}$ for $\alpha < \kappa, \beta < \delta$. (Here $2 \times \alpha$ is the multiplication as ordinals.) Then V_f is a reduced word of appearance $n+1$ in $\mathcal{W}(G_{\alpha\beta}: \alpha < \kappa, \beta < \delta)$ and $\langle \{V_f: f \in \prod_{\beta<\delta} \mu_\beta\}\rangle = *_{f\in\prod_{\beta<\delta}\mu_\beta} \langle V_f\rangle$. A similar argument using V_f's as in the case of Bd_1 shows that $Bd_{n+1} \simeq Bd_n * *_\lambda \mathbb{Z}$.

5. PROOF OF THEOREM 1.3

Answering the second author's question, S. Shelah and L. Struengmann [7] proved that there exists a homomorphism from $\mathbb{X}_\kappa \mathbb{Z}$ to $\mathbb{Z}$ without a finite support for an uncountable κ. Our proof is a modification of their proof.

The largest and smallest elements of $D(L)$ are denoted by ∞ and $-\infty$, respectively. For $d, d' \in D(\overline{W})$ with $d < d'$, $W \restriction (d, d')$ is a subword of W and $W \cong W \restriction (-\infty, d) W \restriction (d, d') W \restriction (d', \infty)$ holds.

For a word V and $d \in D(\overline{V})$, $d \in +(W)$ if $V \restriction (-\infty, d)$ is tail-equivalent to W and $d \in -(W)$ if $V \restriction (d, \infty)$ is head-equivalent to W^{-1}.

The next proof heavily depends on the uncountable cofinality and we refer the reader to [6]. The least uncountable ordinal is denoted by ω_1 and the fact that the cofinality of ω_1 is uncountable is essential later.

Proof (Theorem 1.3). We may assume that I is uncountable, $I_1 = I$, G_i's are nontrivial, and even that $|G_i| \geq 3$ for each i by Lemma 2.5. We apply Proposition 6.2 to the case that $\kappa = \omega_1$ and $X_i = G_i \setminus \{e\}$ $(i \in I)$. By some modification for $G_{\alpha 0}$'s we have

- the index set I contains $\omega_1 \times \delta$ and $|G_{0\beta}| \leq |G_{\alpha\beta}|$ for $\alpha < \omega_1$ and $\beta < \delta$;
- μ_β is an infinite cardinal or $\mu_\beta = 2$, and $|G_{0\beta}| = \mu_\beta$ when μ_β is infinite;
- $G_{\alpha\beta} \cap G_{\alpha'\beta'} = \{e\}$ for $(\alpha, \beta) \neq (\alpha', \beta')$;
- $|\prod_{i\in I} G_i| = |\prod_{0<\beta<\delta} \mu_\beta|$;
- $\{a_{\alpha\gamma}: \gamma < \mu_\beta\} \subseteq G_{\alpha\beta} \setminus \{e\}$ and $a_{\alpha\gamma} \neq a_{\alpha\gamma'}$ for $\gamma \neq \gamma'$.

We regard $\omega_1 \times \delta$ lexicographically ordered.

For $f \in \prod_{0<\beta<\delta} \mu_\beta$, let $\overline{W_f} = \omega_1 \times \delta$, $W_f(\alpha, 0) = a_{\alpha 0}$, and $W_f(\alpha, \gamma) = a_{\alpha f(\gamma)}$ for $0 < \gamma$. Note that $a_{\alpha 0}$'s $(\alpha < \omega_1)$ cofinally appear in W_f; i.e., for each $u \in W_f$ there exists $\alpha_0 < \omega_1$ such that $u < (\alpha, 0)$ for any $\alpha \geq \alpha_0$ and consequently $a_{\alpha 0}$ appears more on the right-hand side than u. Now W_f is a reduced word in $\mathscr{W}(G_{\alpha\beta}\colon \alpha < \omega_1, \beta < \delta)$. To apply Lemma 4.4, let $p\colon \mathbb{X}_{\alpha<\omega_1, \beta<\delta} G_{\alpha\beta} \to \mathbb{X}_{\alpha<\omega_1, 0<\beta<\delta} G_{\alpha\beta}$ be the projection. Then $\langle\{p(W_f)\colon f \in \prod_{0<\beta<\delta} \mu_\beta\}\rangle = *_{f\in\prod_{0<\beta<\delta}\mu_\beta} \langle p(W_f)\rangle$ is a free subgroup by Lemma 4.4. Since p is injective on $\langle\{W_f\colon f \in \prod_{0<\beta<\delta} \mu_\beta\}\rangle$, $\langle\{W_f\colon f \in \prod_{0<\beta<\delta} \mu_\beta\}\rangle = *_{f\in\prod_{0<\beta<\delta}\mu_\beta} \langle W_f\rangle$.

Here we claim that $\{d \in D(\overline{V})\colon d \in +(W_f)$ for some $f\}$ and $\{d \in D(\overline{V})\colon d \in -(W_f)$ for some $f\}$ are finite for a word V. To see this, suppose the negation. Then there are infinitely many $d \in D(\overline{V})$ which are in $+(W_f)$ or $-(W_f)$ for some f. Since the cofinalities of $\overline{W_f}$'s are ω_1, $a_{\alpha 0}$ or $a_{\alpha 0}^{-1}$ appear infinitely many times for sufficiently large $\alpha < \omega_1$, which contradicts that V is a word.

For a reduced word V, let $d_1 \prec \cdots \prec d_n$ be the enumeration of $\{d \in V\colon d \in +(W_f)$ for some $f\} \cup \{d \in V\colon d \in -(W_f)$ for some $f\}$ in the ordering of $D(\overline{V})$. Define x_i ($1 \leq i \leq 2n$) as follows:

$$x_{2i-1} = \begin{cases} W_f & \text{if } d_i \in +(W_f), \\ e & \text{otherwise, i.e. } d_i \notin +(W_f) \quad \text{for any } f \end{cases}$$

$$x_{2i} = \begin{cases} W_f^{-1} & \text{if } d_i \in -(W_f), \\ e & \text{otherwise, i.e. } d_i \notin -(W_f) \quad \text{for any } f. \end{cases}$$

Then define $h\colon \mathbb{X}_{\alpha<\delta} G_\alpha \to *_{f\in\prod_{0<\beta<\delta}\mu_\beta} \langle W_f\rangle$ by: $h(V) = x_1 x_2 \cdots x_{2n}$. To see that h is a homomorphism—i.e., $h(uv) = h(u)h(v)$—let U and V be reduced words for u and v, respectively. Then there exist reduced words X, Y, Z such that $U \cong XY$ and $V \cong Y^{-1}Z$ hold and XZ is quasi-reduced. Since $h(Y^{-1}) = h(Y)^{-1}$ and $h(uv) = h(X)h(Z)$, $h(uv) = h(u)h(v)$ holds. The surjectivity of h is clear and the proof is completed, since $|\mathbb{X}_{i\in I} G_i| = |\prod_{0<\beta<\delta}\mu_\beta|$ holds. ∎

REMARK 5.1. (1) As we noted, the uncountability is essential to Theorem 1.3 by the Higman theorem. The crucial use of the uncountability in the proof is $(\alpha, 0)$ for $\alpha < \omega_1$. An element in Sc_{ω_1} corresponds to a reduced word of countable length and $W_f \notin Sc_{\omega_1}$ for any f. However, W_f belongs to the both Sc and Bd.

(2) As S. Shelah and L. Struengmann [7] showed and we stated in Theorem 1.3, the Higman theorem for $\mathbb{X}_I \mathbb{Z}$ fails in the uncountable case. On the other hand, K. Eda and S. Shelah [4] proved that the Higman theorem for the inverse limit $\varprojlim(*_X\mathbb{Z}, p_{XY}\colon X \subseteq Y \Subset I)$ holds also in the uncountable case. To state it more precisely, let I be an uncountable index set of the cardinality less than the least measurable cardinal. Then for each homomorphism $h\colon \varprojlim(*_{i\in X} G_i, p_{XY}\colon X \subseteq Y \Subset I) \to *_J\mathbb{Z}$ there exists a finite subset F of I and a homomorphism $\bar{h}\colon *_{i\in F} G_i \to *_J Z$ such

that $h = \bar{h} \cdot p_F$, where $p_F\colon \varprojlim(*_{i\in X}G_i, p_{XY}\colon X \subseteq Y \Subset I) \to *_{i\in F}G_i$ is the projection. Both the free complete product $\mathbb{X}_{i\in I}G_i$ and the inverse limit $\varprojlim(*_{i\in X}G_i, p_{XY}\colon X \subseteq Y \Subset I)$ are noncommutative variations of the direct product $\prod_{i\in I}G_i$ in the commutative case [5] and in the countable case these two show the Specker phenomenon. Shelah *et al.*'s theorems show that in the uncountable case these two notions behave differently concerning the Specker phenomenon.

6. CARDINALITY ARGUMENTS

In this section we prove some technical propositions which are necessary to prove our main theorems. All the proofs are elementary, but need ordinal or cardinal arithmetic arguments. We refer the reader to [6] for notions on ordinals, cardinals, and elementary calculations of them. By working more we can drop the assumption of the regularity of κ in Lemma 6.1 and Proposition 6.2, but it is enough for our purpose.

LEMMA 6.1. *Let κ be a regular cardinal and let $\{X_\alpha\colon \alpha < \kappa\}$ be a family of sets such that $|X_\alpha| \leq |X_\beta|$ for $\alpha \leq \beta < \kappa$. Then there exists an injective map $\psi\colon \kappa \times \kappa \to \kappa$ such that $|X_{\psi(0,\gamma)}| \leq |X_{\psi(\beta,\gamma)}|$ for $\beta, \gamma < \kappa$, and $|\prod_{\alpha<\kappa}X_\alpha| = |\prod_{\gamma<\kappa}X_{\psi(0,\gamma)}|$.*

Proof. By the regularity of κ we can choose an injective map $\psi'\colon \kappa \times \kappa \to \kappa$ so that $\psi'(0,\gamma) < \psi'(0,\delta)$ for $\gamma < \delta$. Since $\psi'(0,\gamma) \geq \gamma$, $|\prod_{\alpha<\kappa}X_\alpha| = |\prod_{\gamma<\kappa}X_{\psi'(0,\gamma)}|$. Fix $\gamma < \kappa$. Since the cardinality of $\{\alpha < \kappa\colon |X_\alpha| < |X_{\psi'(0,\gamma)}|\}$ is less than κ, by deleting $\{(\beta,\gamma)\colon |X_{\psi'(\beta,\gamma)}| < |X_{\psi'(0,\gamma)}|\}$ and rearranging it we have the desired ψ. ∎

PROPOSITION 6.2. *Let κ be a regular cardinal and let $\{X_i\colon i \in I\}$ be a family of sets such that $|X_j| \geq 2$ and let $|\{i \in I\colon |X_i| \geq |X_j|\}| \geq \kappa$ for each $j \in I$. Then there exist cardinals μ_β $(\beta < \nu)$ and an injective map $\varphi\colon \kappa \times \delta \to I$ such that $\mu_\beta \leq |X_{\varphi(\alpha,\beta)}|$ for $\alpha < \kappa, \beta < \delta$ and $|\prod_{i\in I}X_i| = |\prod_{\beta<\delta}\mu_\beta|$.*

Proof. First well order the index set I so that $|X_i| \leq |X_j|$ for $i \leq j$. Since we consider the order type of I, we assume I is an ordinal δ. Let $\delta = \kappa \cdot \mu + \delta_0$ for some $0 \leq \delta_0 < \kappa$. (Here, the multiplication and the sum are those for ordinals.) Suppose that $\delta_0 > 0$. Since $\delta_0 < \kappa$, there exists $\alpha_0 < \kappa \cdot \mu$ such that $|X_\alpha| = |X_{\alpha 0}|$ for $\alpha \geq \alpha_0$. Since $|\prod_{\alpha<\delta}X_\alpha| = |\prod_{\alpha<\kappa\cdot\mu}X_\alpha|$, we may assume $\delta = \kappa \cdot \mu$. We apply Lemma 6.1 to $X_{\kappa\cdot\beta+\gamma}$ $(\gamma < \kappa)$ for each $\beta < \mu$ and have $\psi_\beta\colon \kappa \times \kappa \to \kappa$. Finally let $\mu_{\kappa\cdot\beta+\gamma} = |X_{\kappa\cdot\beta+\psi_\beta(0,\gamma)}|$ and $\varphi(\alpha, \kappa\cdot\beta+\gamma) = \kappa\cdot\beta + \psi_\beta(\alpha,\gamma)$ for $\alpha < \kappa, \beta < \mu, \gamma < \kappa$. Now it is easy to check that $\varphi\colon \kappa \times \delta \to \delta$ and μ_β's $(\beta < \delta)$ satisfy the required properties. ∎

PROPOSITION 6.3. *The cardinalities of* $\prod_{i\in I} G_i$ *and* $\mathbb{X}_{i\in I} G_i$ *are the same.*

Proof. Since there is a canonical surjection from $\mathbb{X}_{i\in I} G_i$ to $\prod_{i\in I} G_i$, we have $|\mathbb{X}_{i\in I} G_i| \geq |\prod_{i\in I} G_i|$.

To see the converse inequality, fix a linear ordering L with $|L| \leq |I|$. We may assume that I is infinite and every G_i is nontrivial. Let $f\colon L \to I$ be a map such that $f^{-1}\{i\}$ is finite for each $i \in I$. Since $f^{-1}\{i\}$ is finite for each $i \in I$, the cardinality of the set of all words W such that $\overline{W} = L$ and $W(u) \in G_i$ if $f(u) = i$ is equal to or less than $|\prod_{i\in I} G_i|$. The cardinality of the set of all L's is $2^{|I|}$ and the cardinality of the set of all f's is also $2^{|I|}$ and consequently we have $|\mathbb{X}_{i\in I} G_i| \leq |\prod_{i\in I} G_i|$.

REMARK 6.4. In contrast to Proposition 6.3 the cardinality of the inverse limit $\varprojlim(*_{i\in X} G_i, p_{XY}\colon X \subseteq Y \Subset I)$ may be strictly greater than that of $|\prod_{i\in I} G_i|$. To see this, let G_n be a copy of the integer group $\mathbb{Z}$ for $0 < n < \omega$ and let G_0 be a group of the cardinality which is greater than $2^{\aleph_0}$ and a limit cardinal of its cofinality ω. For $a \in \prod_{0<n<\omega}(G_n \backslash \{e\})$ and $f\colon \omega\backslash\{0\} \to G_0\backslash\{e\}$ we define $x(a, f)$ as $x(a, f)(n) = f(1)^{-1}a(1)f(1)\cdots f(n)^{-1}a(n)f(n) \in *_{i\leq n} G_i$ for each $n < \omega$. Then we have $x(a, f) \in \varprojlim(*_{n<\omega} G_n, p_{mn}\colon m \leq n < \omega)$ and consequently $|\varprojlim(*_{n<\omega} G_n, p_{mn}\colon m \leq n < \omega)| \geq |G_0|^{\omega} > |G_0| = \aleph_0^{\omega}|G_0| = |\prod_{n<\omega} G_n|$.

On the other hand, if the cardinalities of G_i's are same, the cardinality of $\varprojlim(*_{i\in X} G_i, p_{XY}\colon X \subseteq Y \Subset I)$ is the same as that of $|\prod_{i\in I} G_i|$.

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